

Module Shifts and Measure Rigidity in Linear Cellular Automata

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Abstract. Suppose \mathcal{R} is a finite commutative ring of prime characteristic, \mathcal{A} is a finite \mathcal{R} -module, $\mathbb{M} := \mathbb{Z}^D \times \mathbb{N}^E$, and Φ is an \mathcal{R} -linear cellular automaton on $\mathcal{A}^{\mathbb{M}}$. If μ is a Φ -invariant measure which is multiply σ -mixing in a certain way, then we show that μ must be the Haar measure on a coset of some submodule shift of $\mathcal{A}^{\mathbb{M}}$. Under certain conditions, this means μ must be the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{M}}$.

Let \mathcal{A} be a finite set. Let $\mathbb{M} := \mathbb{Z}^D \times \mathbb{N}^E$ be a $(D+E)$ -dimensional lattice, for some $D, E \in \mathbb{N}$, and let $\mathcal{A}^{\mathbb{M}}$ denote the set of all functions $\mathbf{a} : \mathbb{M} \rightarrow \mathcal{A}$, which we regard as \mathbb{M} -indexed *configurations* of elements in \mathcal{A} . We write such a configuration as $\mathbf{a} = [a_{\mathbf{m}}]_{\mathbf{m} \in \mathbb{M}}$, where $a_{\mathbf{m}} \in \mathcal{A}$ for all $\mathbf{m} \in \mathbb{M}$. Treat \mathcal{A} as a discrete topological space; then $\mathcal{A}^{\mathbb{M}}$ is a Cantor space —i.e. it is compact, perfect, totally disconnected, and metrizable.

If $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ and $\mathbb{U} \subset \mathbb{M}$, then we define $\mathbf{a}_{\mathbb{U}} \in \mathcal{A}^{\mathbb{U}}$ by $\mathbf{a}_{\mathbb{U}} := [a_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{U}}$. If $\mathbf{m} \in \mathbb{M}$, then strictly speaking, $\mathbf{a}_{\mathbf{m}+\mathbb{U}} \in \mathcal{A}^{\mathbf{m}+\mathbb{U}}$; however, it will often be convenient to ‘abuse notation’ and treat $\mathbf{a}_{\mathbf{m}+\mathbb{U}}$ as an element of $\mathcal{A}^{\mathbb{U}}$ in the obvious way. Let $\mathbb{H} \subset \mathbb{M}$ be some finite subset, and let $\phi : \mathcal{A}^{\mathbb{H}} \rightarrow \mathcal{A}$ be a function (called a *local rule*). The *cellular automaton* (CA) determined by ϕ is the function $\Phi : \mathcal{A}^{\mathbb{M}} \rightarrow \mathcal{A}^{\mathbb{M}}$ defined by $\Phi(\mathbf{a})_{\mathbf{m}} = \phi(\mathbf{a}_{\mathbf{m}+\mathbb{H}})$ for all $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ and $\mathbf{m} \in \mathbb{M}$. We refer to \mathbb{H} as the *neighbourhood* of Φ .

We will prove a new ‘measure rigidity’ result for linear CA: if Φ is a linear CA and μ is a Φ -invariant measure which is multiply σ -mixing in a certain way, then μ must be the Haar measure on a coset of some submodule shift of $\mathcal{A}^{\mathbb{M}}$. In particular, if $\mathcal{A}^{\mathbb{M}}$ admits no proper mixing subgroup shifts (e.g. $D = 1$ and $\mathcal{A} = \mathbb{Z}/p$, for p prime), then μ must be the uniform measure on $\mathcal{A}^{\mathbb{M}}$. This result is complementary to previous rigidity results of [Sch95b, HMM03, Piv05, Ein05, Sab07].

Terminology & Notation. Throughout, lowercase bold-faced letters ($\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$) denote elements of $\mathcal{A}^{\mathbb{M}}$, and Roman letters (a, b, c, \dots) are elements of \mathcal{A} or ordinary numbers. Lowercase sans-serif ($\dots, \mathbf{m}, \mathbf{n}, \mathbf{p}$) are elements of \mathbb{M} , and upper-case hollow font ($\mathbb{U}, \mathbb{V}, \mathbb{W}, \dots$) are subsets of \mathbb{M} . For any $\mathbf{v} \in \mathbb{M}$, let $\sigma^{\mathbf{v}} : \mathcal{A}^{\mathbb{M}} \rightarrow \mathcal{A}^{\mathbb{M}}$ be the *shift map* defined by $\sigma^{\mathbf{v}}(\mathbf{a})_{\mathbf{m}} = a_{\mathbf{m}+\mathbf{v}}$ for all $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ and $\mathbf{m} \in \mathbb{M}$. Let $\text{CA}(\mathcal{A}^{\mathbb{M}})$ denote the set of cellular automata on $\mathcal{A}^{\mathbb{M}}$; then $\text{CA}(\mathcal{A}^{\mathbb{M}})$ is also the set of continuous transformations of $\mathcal{A}^{\mathbb{M}}$ which commute with all shifts [Hed69, Theorem 3.4]. A *subshift* is a closed subset $\mathbf{S} \subseteq \mathcal{A}^{\mathbb{M}}$ which is invariant under all shifts. Let $\text{CA}(\mathbf{S}) := \{\Phi \in \text{CA}(\mathcal{A}^{\mathbb{M}}) ; \Phi(\mathbf{S}) \subseteq \mathbf{S}\}$.

If $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$, and $\mathbb{K} \subset \mathbb{M}$, recall that $\mathbf{a}_{\mathbb{K}} := [a_{\mathbf{k}}]_{\mathbf{k} \in \mathbb{K}} \in \mathcal{A}^{\mathbb{K}}$. If $\mathbf{S} \subseteq \mathcal{A}^{\mathbb{M}}$ is a subshift, let $\mathbf{S}_{\mathbb{K}} := \{\mathbf{s}_{\mathbb{K}} ; \mathbf{s} \in \mathbf{S}\} \subseteq \mathcal{A}^{\mathbb{K}}$. If $\mathbf{k} \in \mathcal{A}^{\mathbb{K}}$, then let $\langle \mathbf{k} \rangle := \{\mathbf{a} \in \mathcal{A}^{\mathbb{M}} ; \mathbf{a}_{\mathbb{K}} = \mathbf{k}\}$ be the *cylinder set*

defined by \mathbf{k} . The topology (and hence, the Borel sigma-algebra) of $\mathcal{A}^{\mathbb{M}}$ is generated by the collection of all such cylinder sets for all finite $\mathbb{K} \subset \mathbb{M}$. Let $\mathfrak{M}_{\text{cas}}(\mathcal{A}^{\mathbb{M}})$ [resp. $\mathfrak{M}_{\text{cas}}(\mathbf{S})$] be the set of Borel probability measures on $\mathcal{A}^{\mathbb{M}}$ [resp. \mathbf{S}], and let $\mathfrak{M}_{\text{cas}}(\mathcal{A}^{\mathbb{M}}, \sigma)$ [resp. $\mathfrak{M}_{\text{cas}}(\mathbf{S}, \sigma)$] be the shift-invariant measures on $\mathcal{A}^{\mathbb{M}}$ [resp. \mathbf{S}]. If $\Phi \in \mathbf{CA}(\mathbf{S})$, let $\mathfrak{M}_{\text{cas}}(\mathbf{S}, \Phi)$ be the Φ -invariant measures on \mathbf{S} .

Linear CA. Let $(\mathcal{R}, +, \cdot)$ be a finite ring with unity $1_{\mathcal{R}}$, and let $(\mathcal{A}, +, \cdot)$ be a finite \mathcal{R} -module. If $\Phi \in \mathbf{CA}(\mathcal{A}^{\mathbb{M}})$, then Φ is an \mathcal{R} -linear CA (\mathcal{R} -LCA) if the local rule $\phi : \mathcal{A}^{\mathbb{H}} \rightarrow \mathcal{A}$ has the form

$$\phi(\mathbf{a}_{\mathbb{H}}) := \sum_{\mathbf{h} \in \mathbb{H}} \varphi_{\mathbf{h}} a_{\mathbf{h}}, \quad \forall \mathbf{a}_{\mathbb{H}} \in \mathcal{A}^{\mathbb{H}}, \quad (1)$$

where $\varphi_{\mathbf{h}} \in \mathcal{R} \setminus \{0\}$ for each $\mathbf{h} \in \mathbb{H}$. Let $\mathcal{R}\text{-LCA}(\mathcal{A}^{\mathbb{M}})$ be the set of all \mathcal{R} -linear CA on $\mathcal{A}^{\mathbb{M}}$.

EXAMPLE 1: If $(\mathcal{A}, +)$ is a finite abelian group, and $\mathcal{A}^{\mathbb{M}}$ is treated as a Cartesian product and endowed with componentwise addition, then $(\mathcal{A}^{\mathbb{M}}, +)$ is a compact abelian group. If $\Phi \in \mathbf{CA}(\mathcal{A}^{\mathbb{M}})$, then Φ is an *endomorphich cellular automaton* (ECA) if Φ is also a group homomorphism of $(\mathcal{A}^{\mathbb{M}}, +)$. Let \mathcal{E} be the (noncommutative) ring of all group endomorphisms of $(\mathcal{A}, +)$. Then \mathcal{A} is an \mathcal{E} -module, and any ECA on $\mathcal{A}^{\mathbb{M}}$ is an \mathcal{E} -linear CA. (In the literature, these are often just called *linear CA*.)

(b) In particular, let $m \in \mathbb{N}$ and let $\mathcal{A} = (\mathbb{Z}/m, +)$, with addition modulo m . Then $\mathcal{E} = (\mathbb{Z}/m, +, \cdot)$ [with multiplication modulo m], so eqn.(1) becomes $\phi(\mathbf{a}_{\mathbb{H}}) = \sum_{\mathbf{h} \in \mathbb{H}} \varphi_{\mathbf{h}} a_{\mathbf{h}} \pmod{m}$, where $\varphi_{\mathbf{h}} \in \{1 \dots m\}$ for each $\mathbf{h} \in \mathbb{H}$.

(c) Let $k \in \mathbb{N}$, and let $\mathcal{R} := \mathbb{F}_{p^k}$ be the unique finite field of order p^k [in particular, if $k = 1$ then $\mathbb{F}_{p^k} = \mathbb{Z}/p$ as in Example (b)]. Let \mathcal{A} be any finite-dimensional \mathbb{F}_{p^k} -vector space (e.g. $\mathcal{A} := (\mathbb{F}_{p^k})^m$, for some $m \in \mathbb{N}$); then $\mathcal{A}^{\mathbb{M}}$ is an (infinite-dimensional) \mathbb{F}_{p^k} -vector space, and Φ is an \mathbb{F}_{p^k} -LCA iff Φ is a linear endomorphism of $\mathcal{A}^{\mathbb{M}}$.

(d) Let \mathcal{A} be a finite-dimensional \mathbb{F}_{p^k} -vector space, and let $\text{End}(\mathcal{A})$ be the (noncommutative) ring of all \mathbb{F}_{p^k} -linear endomorphisms of \mathcal{A} . Suppose $\{\varphi_{\mathbf{h}}\}_{\mathbf{h} \in \mathbb{H}} \subset \text{End}(\mathcal{A})$ is a collection of endomorphisms which commute with one another, and let \mathcal{R} be the subring of $\text{End}(\mathcal{A})$ generated by $\{\varphi_{\mathbf{h}}\}_{\mathbf{h} \in \mathbb{H}}$; then \mathcal{R} is a commutative ring, \mathcal{A} is an \mathcal{R} -module, and if Φ is as in eqn.(1), then Φ is an \mathcal{R} -LCA. \diamond

Example 1(a) is ‘universal’ in the following sense: any \mathcal{R} -module is also an abelian group, and any \mathcal{R} -linear CA is automatically an ECA. However, in a general ECA, the coefficients $\{\varphi_{\mathbf{h}}\}_{\mathbf{h} \in \mathbb{H}} \subset \mathcal{E}$ do not commute, because the endomorphism ring \mathcal{E} is not commutative unless $\mathcal{A} = \mathbb{Z}/m$, as in Example 1(b). If \mathcal{R} is commutative, then \mathcal{R} -LCA are much easier to analyze than general ECA.

If $r \in \mathcal{R}$, then the *characteristic* of r is the smallest $m \in \mathbb{N}$ such that $m \cdot r = 0$ (or it is 0 if there is no such m). The *characteristic* of \mathcal{R} is the characteristic of the unity element $1_{\mathcal{R}}$. (For example, $\mathcal{R} = \mathbb{Z}/m$ has characteristic m .) If \mathcal{R} has characteristic m , then $m \cdot r = 0$ for all $r \in \mathcal{R}$; hence the characteristic of r divides m . We will be mainly interested in the case when \mathcal{R} is a *commutative* ring of prime characteristic, as in Examples 1(c,d).

Subgroup shifts and submodule shifts. Suppose $(\mathcal{A}, +)$ is a finite abelian group, so that $(\mathcal{A}^{\mathbb{M}}, +)$ is compact abelian. A *subgroup shift* is a closed, shift-invariant subgroup $\mathbf{G} \subset \mathcal{A}^{\mathbb{M}}$ (i.e. \mathbf{G} is both a subshift and a subgroup); see [Kit87, Kit00, KS89, KS92, Sch95a]. If \mathcal{A} is an \mathcal{R} -module, then $\mathcal{A}^{\mathbb{M}}$ is also an \mathcal{R} -module under componentwise \mathcal{R} -multiplication. An \mathcal{R} -*submodule shift* is a subgroup shift which is also an \mathcal{R} -submodule. For example, if $\Phi \in \mathcal{R}\text{-LCA}(\mathcal{A}^{\mathbb{M}})$, then $\Phi(\mathcal{A}^{\mathbb{M}})$ and $\ker(\Phi) := \Phi^{-1}\{\mathbf{0}\}$ are submodule shifts (here $\mathbf{0} \in \mathcal{A}^{\mathbb{M}}$

is the constant zero element). Also $\text{Fix}[\Phi] := \{\mathbf{a} \in \mathcal{A}^{\mathbb{M}}; \Phi(\mathbf{a}) = \mathbf{a}\}$ is a submodule shift (because $\text{Fix}[\Phi] = \ker(\Phi - \text{Id})$). If $\mathcal{A} = \mathcal{R} = \mathbb{Z}/m$, then every subgroup shift is a submodule shift, and vice versa. However, in general the \mathcal{R} -submodule shifts form a more restricted class.

To study the ergodic theory of \mathcal{R} -LCA, it is first necessary to characterize their invariant measures. If $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{M}}$ is a subgroup shift, then the *Haar measure* of \mathbf{G} is the unique $\eta_{\mathbf{G}} \in \mathfrak{M}_{\text{cas}}(\mathbf{G})$ which is invariant under translation by all elements of \mathbf{G} . That is, if $\mathbf{g} \in \mathbf{G}$, and $\mathbf{U} \subset \mathbf{G}$ is any measurable subset, and $\mathbf{U} + \mathbf{g} := \{\mathbf{u} + \mathbf{g}; \mathbf{u} \in \mathbf{U}\}$, then $\eta_{\mathbf{G}}[\mathbf{U} + \mathbf{g}] = \eta_{\mathbf{G}}[\mathbf{U}]$. In particular, if $\mathbf{G} = \mathcal{A}^{\mathbb{M}}$, then $\eta_{\mathbf{G}}$ is just the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{M}}$. Let $\text{ECA}(\mathbf{G}) := \{\Phi \in \text{CA}(\mathcal{A}^{\mathbb{M}}); \Phi \text{ is an ECA and } \Phi(\mathbf{G}) \subseteq \mathbf{G}\}$.

PROPOSITION 2. *Let $(\mathcal{A}, +)$ be a finite abelian group, let $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{M}}$ be a subgroup shift, and let $\Phi \in \text{ECA}(\mathbf{G})$. Then $(\Phi(\eta_{\mathbf{G}}) = \eta_{\mathbf{G}}) \iff (\Phi(\mathbf{G}) = \mathbf{G})$.*

Proof: (a) “ \implies ” is because $\mathbf{G} = \text{supp}(\eta_{\mathbf{G}})$. To see “ \impliedby ”, note that $\Phi(\eta_{\mathbf{G}}) \in \mathfrak{M}_{\text{cas}}(\mathbf{G})$. Thus, it suffices to show that $\Phi(\eta_{\mathbf{G}})$ is invariant under all \mathbf{G} -translations. Let $\mathbf{g} \in \mathbf{G}$, and let $\tau^{\mathbf{g}} : \mathbf{G} \rightarrow \mathbf{G}$ be the translation map [i.e. $\tau^{\mathbf{g}}(\mathbf{h}) := \mathbf{g} + \mathbf{h}$]. Find $\mathbf{h} \in \mathbf{G}$ such that $\Phi(\mathbf{h}) = \mathbf{g}$ (this \mathbf{h} exists because $\Phi(\mathbf{G}) = \mathbf{G}$). Then $\tau^{\mathbf{g}}[\Phi(\eta_{\mathbf{G}})] \stackrel{(*)}{=} \Phi[\tau^{\mathbf{h}}(\eta_{\mathbf{G}})] \stackrel{(\dagger)}{=} \Phi(\eta_{\mathbf{G}})$, where $(*)$ is because $\tau^{\mathbf{g}} \circ \Phi = \Phi \circ \tau^{\mathbf{h}}$, and (\dagger) is because $\eta_{\mathbf{G}}$ is the Haar measure. This holds for all $\mathbf{g} \in \mathbf{G}$. But $\eta_{\mathbf{G}}$ is the unique probability measure on \mathbf{G} such that $\tau^{\mathbf{g}}(\eta_{\mathbf{G}}) = \eta_{\mathbf{G}}$ for all $\mathbf{g} \in \mathbf{G}$; thus $\Phi(\eta_{\mathbf{G}}) = \eta_{\mathbf{G}}$. \square

Some ECA exhibit a great deal of *measure rigidity*, meaning that the Haar measures of Φ -invariant subgroup shifts are the *only* Φ -invariant measures satisfying certain ‘nondegeneracy’ conditions. For example, Host, Maass and Martínez showed that, if p is prime and $\mathcal{A} = \mathbb{Z}/p$, and Φ is a nearest neighbour \mathbb{Z}/p -LCA on $\mathcal{A}^{\mathbb{Z}}$, then the only positive-entropy, σ -ergodic, Φ -invariant measure is the Haar measure on $\mathcal{A}^{\mathbb{Z}}$ —i.e. the uniform Bernoulli measure [HMM03, Thm.12]. This result was vastly generalized by Sablik, who showed that, if $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{Z}}$ is any subgroup shift and $\Phi \in \text{ECA}(\mathbf{G})$, then the only positive-entropy, Φ -invariant measure satisfying certain ergodicity conditions is the Haar measure on \mathbf{G} [Sab07, Thm. 3.3 and 3.4]. (Actually, Sablik’s result is even more general, since it allows any abelian group shift structure on $\mathcal{A}^{\mathbb{Z}}$.) See also [Piv05] for similar results concerning ECA in the case when $\mathcal{A}^{\mathbb{Z}}$ is a *nonabelian* group shift, as well as *multiplicative* CA (in the case when (\mathcal{A}, \cdot) is a nonabelian group).

All of these results are for *one-dimensional* ECA. Einsiedler [Ein05, Corollary 2.3] has a similar rigidity result for automorphic \mathbb{Z}^D -actions on compact abelian groups (e.g. the \mathbb{Z}^D -shift action on a subgroup shift $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{Z}^D}$). This theorem can be easily translated into equivalent rigidity results for ECA in $\mathcal{A}^{\mathbb{Z}^{D-1}}$. Like [HMM03] and [Sab07], Einsiedler requires both an entropy condition and fairly strong ergodicity hypotheses.

Let \mathcal{R} be a commutative ring of characteristic p . We will prove a measure rigidity result for multidimensional \mathcal{R} -LCA whose only requirement on the measure is a limited form of multiple mixing. Our result is philosophically similar to the rigidity results in [Sch95b] or [Sch95a, §29], but it is applicable to much larger class of cellular automata.

Let $\mu \in \mathfrak{M}_{\text{cas}}(\mathcal{A}^{\mathbb{M}}; \sigma)$. For any $H \in \mathbb{N}$, we say $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is (σ, H) -*mixing* if, for any Borel measurable $\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_H \subseteq \mathcal{A}^{\mathbb{M}}$.

$$\lim_{\substack{m_0, m_1, \dots, m_H \in \mathbb{M} \\ |m_h - m_i| \rightarrow \infty \\ \forall h \neq i \in [0 \dots H]}} \mu \left[\bigcap_{h=0}^H \sigma^{-m_h}(\mathbf{B}_h) \right] = \prod_{h=0}^H \mu[\mathbf{B}_h].$$

If $\mathbb{H} \subset \mathbb{M}$ is a finite subset, then μ is \mathbb{H} -*mixing* if, for any finite subset $\mathbb{B} \subset \mathbb{M}$ and any \mathbb{H} -indexed collection of \mathbb{B} -words $\{\mathbf{b}_h\}_{h \in \mathbb{H}} \subset \mathcal{A}^{\mathbb{B}}$, with cylinder sets $\mathbf{B}_h := \langle \mathbf{b}_h \rangle \subset \mathcal{A}^{\mathbb{M}}$ for all $h \in \mathbb{H}$, we have

$$\lim_{n \rightarrow \infty} \mu \left[\bigcap_{h \in \mathbb{H}} \sigma^{-nh}(\mathbf{B}_h) \right] = \prod_{h \in \mathbb{H}} \mu[\mathbf{B}_h]. \quad (2)$$

For example, if $|\mathbb{H}| = H$, then any (σ, H) -mixing measure is \mathbb{H} -mixing. In particular, any nontrivial Bernoulli measure in $\mathcal{A}^{\mathbb{M}}$ is \mathbb{H} -mixing, and any mixing Markov measure in $\mathcal{A}^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{N}}$ is \mathbb{H} -mixing.

If $\mathbf{S} \subseteq \mathcal{A}^{\mathbb{M}}$ is a subshift, then \mathbf{S} is *topologically \mathbb{H} -mixing* if, for any finite $\mathbb{B} \subset \mathbb{M}$ and \mathbb{H} -indexed collection of \mathbb{B} -words $\{\mathbf{b}_h\}_{h \in \mathbb{H}} \subseteq \mathbf{S}_{\mathbb{B}}$, with $\mathbf{B}_h := \langle \mathbf{b}_h \rangle$ as above, there is some $N \in \mathbb{N}$ such that, for all $n > N$, we have $\bigcap_{h \in \mathbb{H}} \sigma^{-nh}(\mathbf{B}_h) \neq \emptyset$. For example, if μ is a \mathbb{H} -mixing measure, then $\mathbf{S} = \text{supp}(\mu)$ is a topologically \mathbb{H} -mixing subshift. In particular, any irreducible Markov subshift of $\mathcal{A}^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{N}}$ is topologically \mathbb{H} -mixing.

A *coset shift* is a subshift \mathbf{C} which is a coset of some submodule shift $\mathbf{S} \subset \mathcal{A}^{\mathbb{M}}$. For example, for any $c \in \mathcal{A}$, let $c^{\mathbb{M}} \in \mathcal{A}^{\mathbb{M}}$ denote the constant configuration equal to c everywhere. Then $c^{\mathbb{M}} + \mathbf{S}$ is a coset shift. If $\mathbf{C} \subseteq \mathcal{A}^{\mathbb{M}}$ is any subshift, and $\mathbf{C} - \mathbf{C} := \{\mathbf{c} - \mathbf{c}' ; \mathbf{c}, \mathbf{c}' \in \mathbf{C}\}$, then it is easy to see that

$$(\mathbf{C} \text{ is a coset shift}) \iff (\mathbf{C} - \mathbf{C} \text{ is a submodule shift}). \quad (3)$$

Let $\mathbf{c} \in \mathcal{A}^{\mathbb{M}}$, and suppose $\mathbf{C} := \mathbf{c} + \mathbf{S}$ is a coset shift. If $\tau^{\mathbf{c}} : \mathcal{A}^{\mathbb{M}} \rightarrow \mathcal{A}^{\mathbb{M}}$ is the translation map $\tau^{\mathbf{c}}(\mathbf{a}) := \mathbf{a} + \mathbf{c}$, then the *Haar measure* of \mathbf{C} is defined then $\eta_{\mathbf{C}} := \tau^{\mathbf{c}}(\eta_{\mathbf{S}})$, where $\eta_{\mathbf{S}}$ is the Haar measure of \mathbf{S} . (This definition is independent of the choice of $\mathbf{c} \in \mathbf{C}$.)

If $0 \neq \varphi \in \mathcal{R}$, then an \mathcal{R} -module \mathcal{A} is φ -*torsion-free* if $\varphi a \neq 0$ for all $a \in \mathcal{A} \setminus \{0\}$. Say φ is a *unit* if it has a multiplicative inverse in \mathcal{R} . This means that the function $\mathcal{A} \ni a \mapsto \varphi a \in \mathcal{A}$ is a group automorphism of $(\mathcal{A}, +)$. If $\mathcal{R} = \mathbb{Z}/m$ [Example 1(b)], then φ is a unit if and only if φ is coprime to m . If \mathcal{R} is a field [Example 1(c)], then every nonzero element of \mathcal{R} is a unit. If φ is a unit, then every \mathcal{R} -module is φ -torsion free.

Suppose Φ has local rule (1) and \mathcal{R} has characteristic p . For all $j \in \mathbb{N}$, let \mathcal{R}_j be the subring of \mathcal{R} generated by $\{\varphi_h^{p^j}\}_{h \in \mathbb{H}}$. This yields a descending chain $\mathcal{R} \supseteq \mathcal{R}_1 \supseteq \mathcal{R}_2 \supseteq \dots$ of finite rings (because \mathcal{R} is finite), so there is some $J \in \mathbb{N}$ such that $\mathcal{R}_J = \mathcal{R}_{J+1} = \dots = \bigcap_{j=1}^{\infty} \mathcal{R}_j$. Let $\mathcal{R}_{\Phi} := \mathcal{R}_J$. For any $k \in \mathbb{N}$, let $\overline{\varphi}_k := (\sum_{h \in \mathbb{H}} \varphi_h^{p^k}) - 1$; this defines a sequence $\{\overline{\varphi}_k\}_{k=1}^{\infty} \subseteq \mathcal{R}$ which is eventually periodic (because \mathcal{R} is finite); thus, there is a nonempty set $\mathcal{F}_{\Phi} \subseteq \mathcal{R}$ of elements which appear infinitely often in $\{\overline{\varphi}_k\}_{k=1}^{\infty}$. We now come to our main result.

THEOREM 3. *Let \mathcal{R} be a finite commutative ring of prime characteristic p , let \mathcal{A} be a finite \mathcal{R} -module, and let $\Phi \in \mathcal{R}\text{-LCA}(\mathcal{A}^{\mathbb{M}})$ have local rule (1), with $|\mathbb{H}| \geq 2$.*

- (a) *Suppose φ_h is a unit for some $h \in \mathbb{H}$. If \mathbf{C} is a topologically \mathbb{H} -mixing, Φ -invariant subshift of $\mathcal{A}^{\mathbb{M}}$, then \mathbf{C} is a coset shift of some \mathcal{R}_{Φ} -submodule shift \mathbf{S} .*
- (b) *If there exists $\overline{\varphi} \in \mathcal{F}_{\Phi}$ and some finite $\mathbb{B} \subset \mathbb{M}$ such that $\mathcal{A}^{\mathbb{B}}/\mathbf{S}_{\mathbb{B}}$ is $\overline{\varphi}$ -torsion free (e.g. if $\overline{\varphi}$ is a unit), and \mathbf{C} is as in (a), then actually $\mathbf{C} = \mathbf{S}$.*
- (c) *Suppose φ_h is a unit for every $h \in \mathbb{H}$. If μ is a (Φ, σ) -invariant, \mathbb{H} -mixing measure on $\mathcal{A}^{\mathbb{M}}$, then μ is the Haar measure of a Φ -invariant coset shift \mathbf{C} of some \mathcal{R}_{Φ} -submodule shift \mathbf{S} .*

If the hypothesis of (b) holds, then μ is the Haar measure on \mathbf{S} .

EXAMPLE 4: Let $\mathbb{M} := \mathbb{Z} \times \mathbb{N}$, let $\mathcal{R} = \mathcal{A} := \mathbb{Z}/2$, and let $\mathbb{B} := \{(0, -1); (0, 0); (0, 1); (1, 0)\}$; then $\mathbf{S} := \{\mathbf{a} \in \mathcal{A}^{\mathbb{M}} ; \sum_{b \in \mathbb{B}} a_{b+m} = 0, \forall m \in \mathbb{M}\}$ is a submodule shift. To visualize $\eta_{\mathbf{S}}$, note

that any $\mathbf{s} \in \mathbf{S}$ is entirely determined by its ‘zeroth row’ $\mathbf{s}_{\mathbb{Z} \times \{0\}}$; this yields a bijection $\Psi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathbf{S}$, and $\eta_{\mathbf{S}} = \Psi(\eta)$, where η is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$.

If $\mathbb{H} := \{(0, 0); (0, 1); (1, 0)\}$, then \mathbf{S} and $\eta_{\mathbf{S}}$ are \mathbb{H} -mixing. Let Φ be the LCA with local rule $\phi(\mathbf{a}_{\mathbb{H}}) = \sum_{\mathbf{h} \in \mathbb{H}} a_{\mathbf{h}}$ (i.e. $\varphi_{\mathbf{h}} = 1$ for all $\mathbf{h} \in \mathbb{H}$). Then $\Phi(\mathbf{S}) = \mathbf{S}$, so $\Phi(\eta_{\mathbf{S}}) = \eta_{\mathbf{S}}$, by Proposition 2. However, $\overline{\varphi}_k = |\mathbb{H}| + 1 = 4 \equiv 0 \pmod{2}$ for all $k \in \mathbb{N}$, so the ‘torsion-free’ condition of Theorem 3(b) is never satisfied; thus, nontrivial Φ -invariant coset shifts might exist. Indeed, let $\mathbf{c} \in \mathcal{A}^{\mathbb{M}}$ be the ‘checkerboard’ configuration defined by $c_{m,n} := (m+n) \pmod{2}$, for all $(m, n) \in \mathbb{M}$; then $\mathbf{c} \notin \mathbf{S}$, and $\mathbf{C} := \mathbf{c} + \mathbf{S}$ is a nontrivial coset shift of \mathbf{S} . Furthermore, $\Phi(\mathbf{c}) = \mathbf{c}$, so $\Phi(\mathbf{C}) = \mathbf{C}$; thus, \mathbf{C} is an \mathbb{H} -mixing, Φ -invariant coset shift, as in Theorem 3(a), while $\eta_{\mathbf{C}}$ is an \mathbb{H} -mixing, Φ -invariant measure, as in Theorem 3(c). \diamond

COROLLARY 5. *Let $\mathcal{A} = \mathcal{R} := \mathbb{Z}/p$, where p is prime. Let $\mathbb{M} := \mathbb{Z}^D \times \mathbb{N}^E$, and let $\Phi \in \mathbb{Z}/p\text{-LCA}(\mathcal{A}^{\mathbb{M}})$ have a neighbourhood of cardinality $H \geq 2$. Let $\mu \in \mathfrak{M}_{\text{cas}}(\mathcal{A}^{\mathbb{M}}; \Phi, \sigma)$ be (σ, H) -mixing. Suppose that either*

[i] $D+E = 1$; or [ii] $h(\mu, \sigma) > 0$ (and $D+E \geq 1$).

Then μ is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{M}}$.

Proof: Every nonzero element of the field \mathbb{Z}/p is a unit, so we can use Theorem 3(c). Case [i] is because $(\mathbb{Z}/p)^{\mathbb{Z}}$ and $(\mathbb{Z}/p)^{\mathbb{N}}$ have no proper infinite subgroup shifts (because if \mathbf{S} was such a subgroup shift, then $\{a \in \mathbb{Z}/p; [a, 0] \in \mathbf{S}_{\{0,1\}}\}$ would be a proper nontrivial subgroup of \mathbb{Z}/p , which is impossible). Case [ii] is because $(\mathbb{Z}/p)^{\mathbb{Z}^D \times \mathbb{N}^E}$ has no proper subgroup shifts of nonzero entropy [Sch95a, first paragraph of §25, p.228]. \square

Theorem 3(c) is somewhat similar to [Sch95b] or [Sch95a, Corollary 29.5, p.289], which characterizes the σ -invariant measures of a subgroup shift $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{M}}$. However, Schmidt requires \mathbf{G} to be ‘almost minimal’ (i.e. to have no infinite σ -invariant subgroups), whereas we do not.

To prove Theorem 3, we use tools from number theory and harmonic analysis. If $\mathbb{M} := \mathbb{Z}^D \times \mathbb{N}^E$, then any \mathcal{R} -linear CA on $\mathcal{A}^{\mathbb{M}}$ can be written as a ‘Laurent polynomial of shifts’ with \mathcal{R} -coefficients. That is, if Φ has local rule (1), then for any $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$,

$$\Phi(\mathbf{a}) := \sum_{\mathbf{h} \in \mathbb{H}} \varphi_{\mathbf{h}} \sigma^{\mathbf{h}}(\mathbf{a}) \quad (\text{where we add configurations componentwise}). \quad (4)$$

We indicate this by writing “ $\Phi = F(\sigma)$ ”, where $F \in \mathcal{R}[x_1^{\pm 1}, \dots, x_D^{\pm 1}; y_1, \dots, y_E]$ is the $(D+E)$ -variable Laurent polynomial defined:

$$F(x_1, \dots, x_D; y_1, \dots, y_E) := \sum_{(h_1, \dots, h_D; h'_1, \dots, h'_E) \in \mathbb{H}} \varphi_{\mathbf{h}} x_1^{h_1} \dots x_D^{h_D} y_1^{h'_1} \dots y_E^{h'_E}.$$

If F and G are two such polynomials, and $\Phi = F(\sigma)$ while $\Gamma = G(\sigma)$, then $\Phi \circ \Gamma = (F \cdot G)(\sigma)$, where $F \cdot G$ is the product of F and G in the polynomial ring $\mathcal{R}[x_1^{\pm 1}, \dots, x_D^{\pm 1}; y_1, \dots, y_E]$. In particular, this means that $\Phi^t = F^t(\sigma)$ for all $t \in \mathbb{N}$. Thus, iterating an \mathcal{R} -LCA is equivalent to computing the powers of a polynomial. If \mathcal{R} is commutative, we can do this with the Binomial Theorem, and if \mathcal{R} has characteristic p , then we can compute the binomial coefficients modulo p . In particular, if p is prime, and Φ has polynomial representation (4), then for any $k \in \mathbb{N}$, Fermat’s Little Theorem implies:

$$\Phi^{p^k} = \sum_{\mathbf{h} \in \mathbb{H}} \varphi_{\mathbf{h}}^{p^k} \sigma^{p^k \mathbf{h}}. \quad (5)$$

LEMMA 6. Let \mathcal{R} be a finite ring generated by a set r_1, \dots, r_H , at least one of which is a unit. Let \mathcal{A} be an \mathcal{R} -module and let $\mathbf{S} \subseteq \mathcal{A}^{\mathbb{M}}$ be a subshift containing $\mathbf{0}$. The following are equivalent:

- (a) \mathbf{S} is an \mathcal{R} -submodule shift.
- (b) For any finite $\mathbb{B} \subset \mathbb{M}$, $\mathbf{S}_{\mathbb{B}}$ is an \mathcal{R} -submodule of $\mathcal{A}^{\mathbb{B}}$.
- (c) For any finite $\mathbb{B} \subset \mathbb{M}$ and any $\mathbf{s}_1, \dots, \mathbf{s}_H \in \mathbf{S}_{\mathbb{B}}$, we have $\sum_{h=1}^H r_h \mathbf{s}_h \in \mathbf{S}_{\mathbb{B}}$.

Proof: Clearly (a) \Leftrightarrow (b) \Rightarrow (c). To see that (c) \Rightarrow (b), let $\tilde{\mathcal{R}}$ be the set of \mathbb{N} -linear combinations of products of $\{r_h\}_{h=1}^H$. Then (c) implies that $\tilde{\mathcal{R}}\mathbf{S}_{\mathbb{B}} \subseteq \mathbf{S}_{\mathbb{B}}$ (because $\mathbf{0} \in \mathbf{S}$). Thus it suffices to show that $\tilde{\mathcal{R}} = \mathcal{R}$.

Every element of \mathcal{R} is a \mathbb{Z} -linear combination of products of $\{r_h\}_{h=1}^H$; hence we need only show that $-1_{\mathcal{R}} \in \tilde{\mathcal{R}}$. Some r_h is a unit, and \mathcal{R} is finite, so there exists $n \in \mathbb{N}$ with $1_{\mathcal{R}} = r_h^n \in \tilde{\mathcal{R}}$. But \mathcal{R} has characteristic $c < \infty$, so $-1_{\mathcal{R}} = (c-1)1_{\mathcal{R}} \in \tilde{\mathcal{R}}$. \square

Proof of Theorem 3(a). Let $\mathbf{C} \subset \mathcal{A}^{\mathbb{M}}$ be a topologically \mathbb{H} -mixing, Φ -invariant subshift. Let $\mathbf{S} := \mathbf{C} - \mathbf{C}$; then Fact (3) says that it suffices to show that \mathbf{S} is an \mathcal{R}_{Φ} -submodule shift. Now, \mathbf{S} is a subshift, $\mathbf{0} \in \mathbf{S}$, and for all $k \in \mathbb{N}$, some element of $\{\varphi_h^{p^k}\}_{h \in \mathbb{H}}$ is a unit, so we can use Lemma 6.

Let $\mathbb{B} \subset \mathbb{M}$ be finite. Now, \mathbf{S} is \mathbb{H} -mixing (because \mathbf{C} is), so there exists $N \in \mathbb{N}$ such that, for any $n > N$, and any \mathbb{H} -indexed collection $\{\mathbf{b}_h\}_{h \in \mathbb{H}} \subseteq \mathbf{S}_{\mathbb{B}}$, there exists $\mathbf{s} \in \mathbf{S}$ with

$$\mathbf{s}_{\mathbb{B}+nh} = \mathbf{b}_h, \text{ for all } h \in \mathbb{H}. \quad (6)$$

Make k large enough that $p^k > N$, and such that \mathcal{R}_{Φ} is generated by $\{\varphi_h^{p^k}\}_{h \in \mathbb{H}}$. Let $\{\mathbf{b}_h\}_{h \in \mathbb{H}} \subseteq \mathbf{S}_{\mathbb{B}}$ be arbitrary, and find $\mathbf{s} \in \mathbf{S}$ satisfying eqn.(6) for $n := p^k$. Then

$$\Phi^{p^k}(\mathbf{s})_{\mathbb{B}} \stackrel{(*)}{=} \sum_{h \in \mathbb{H}} \varphi_h^{p^k} \sigma^{p^k h}(\mathbf{s})_{\mathbb{B}} = \sum_{h \in \mathbb{H}} \varphi_h^{p^k} \mathbf{s}_{\mathbb{B}+p^k h} \stackrel{(\dagger)}{=} \sum_{h \in \mathbb{H}} \varphi_h^{p^k} \mathbf{b}_h.$$

[$(*)$ is by eqn.(5) and (\dagger) is by eqn.(6).] But $[\Phi^{p^k}(\mathbf{s})]_{\mathbb{B}} \in \mathbf{S}_{\mathbb{B}}$, because $\Phi(\mathbf{S}) \subseteq \mathbf{S}$, because $\Phi(\mathbf{C}) \subseteq \mathbf{C}$. This verifies condition (c) of Lemma 6 for any finite $\mathbb{B} \subset \mathbb{M}$ and $\{\mathbf{b}_h\}_{h \in \mathbb{H}} \subseteq \mathbf{S}_{\mathbb{B}}$. \square

LEMMA 7. Let \mathcal{R} , \mathcal{A} , Φ , and \mathcal{F}_{Φ} be as in Theorem 3, and let $\mathbf{S} \subset \mathcal{A}^{\mathbb{M}}$ be a submodule shift. If \mathbf{C} is any \mathbb{H} -mixing, Φ -invariant coset shift of \mathbf{S} , then $\overline{\varphi}\mathbf{C} \subseteq \mathbf{S}$ for all $\overline{\varphi} \in \mathcal{F}_{\Phi}$.

Proof: Let $\overline{\varphi} \in \mathcal{F}_{\Phi}$ and let $\mathbf{a} \in \mathbf{C}$. To show that $\overline{\varphi}\mathbf{a} \in \mathbf{S}$, it suffices to show, for any finite $\mathbb{B} \subset \mathbb{M}$, that $\overline{\varphi}\mathbf{a}_{\mathbb{B}} \in \mathbf{S}_{\mathbb{B}}$.

There exist arbitrarily large $k \in \mathbb{N}$ with $\overline{\varphi}_k = \overline{\varphi}$. But if k is large enough, then there exists $\mathbf{c} \in \mathbf{C}$ with $\mathbf{c}_{p^k h + \mathbb{B}} = \mathbf{a}_{\mathbb{B}}$ for all $h \in \mathbb{H}$ (because \mathbf{C} is \mathbb{H} -mixing). Thus, $\Phi^{p^k}(\mathbf{c})_{\mathbb{B}} = (\sum_{h \in \mathbb{H}} \varphi_h^{p^k}) \mathbf{a}_{\mathbb{B}}$, by eqn.(5). Thus, $\overline{\varphi}_k \mathbf{a}_{\mathbb{B}} = \Phi^{p^k}(\mathbf{c})_{\mathbb{B}} - \mathbf{c}_{\mathbb{B}}$. But $\Phi^{p^k}(\mathbf{c})_{\mathbb{B}} - \mathbf{c}_{\mathbb{B}} = (\Phi^{p^k}(\mathbf{c}) - \mathbf{c})_{\mathbb{B}} \in \mathbf{S}_{\mathbb{B}}$ because $\Phi^{p^k}(\mathbf{c}) - \mathbf{c} \stackrel{(*)}{=} \mathbf{c} - \mathbf{c} \stackrel{(\dagger)}{=} \mathbf{S}$, where $(*)$ is because $\Phi^{p^k}(\mathbf{C}) \subseteq \mathbf{C}$, and (\dagger) is by Fact (3). \square

Proof of Theorem 3(b). (by contradiction) If $\mathcal{A}^{\mathbb{B}}/\mathbf{S}_{\mathbb{B}}$ is $\overline{\varphi}$ -torsion free, then $\mathcal{A}^{\mathbb{B}'}/\mathbf{S}_{\mathbb{B}'}$ is also $\overline{\varphi}$ -torsion free, for any $\mathbb{B}' \supseteq \mathbb{B}$. If $\mathbf{C} \neq \mathbf{S}$, and \mathbb{B} is large enough, then $\mathbf{C}_{\mathbb{B}}$ is a nontrivial coset of the submodule $\mathbf{S}_{\mathbb{B}}$ in $\mathcal{A}^{\mathbb{B}}$; thus, $\mathbf{C}_{\mathbb{B}}$ is nontrivial as an element of the quotient module $\mathcal{A}^{\mathbb{B}}/\mathbf{S}_{\mathbb{B}}$. But Lemma 7 implies that $\overline{\varphi}\mathbf{C}_{\mathbb{B}} \subseteq \mathbf{S}_{\mathbb{B}}$, so $\overline{\varphi}$ annihilates $\mathbf{C}_{\mathbb{B}}$ in $\mathcal{A}^{\mathbb{B}}/\mathbf{S}_{\mathbb{B}}$, so $\mathcal{A}^{\mathbb{B}}/\mathbf{S}_{\mathbb{B}}$ has nontrivial $\overline{\varphi}$ -torsion, which is a contradiction. By contradiction, $\mathbf{C} = \mathbf{S}$. \square

Let $\mathbb{T} := \{c \in \mathbb{C} ; |c| = 1\}$ be the unit circle group. If $(\mathbf{G}, +)$ is a compact abelian group (e.g. $\mathbf{G} := \mathcal{A}^{\mathbb{M}}$ where $(\mathcal{A}, +)$ is a finite abelian group), then a *character* on \mathbf{G} is a continuous group homomorphism $\chi : (\mathbf{G}, +) \rightarrow (\mathbb{T}, \cdot)$. Let $\widehat{\mathbf{G}}$ denote the group of characters of \mathbf{G} . If $\mu \in \mathfrak{M}_{\text{cas}}(\mathbf{G})$, then μ is uniquely identified by its *Fourier coefficients*

$$\widehat{\mu}[\chi] := \int_{\mathbf{G}} \chi \, d\mu, \quad \text{for all } \chi \in \widehat{\mathbf{G}}.$$

For example, if $\eta_{\mathbf{G}}$ is the Haar measure on \mathbf{G} , and $\mathbf{1} \in \widehat{\mathbf{G}}$ is the trivial character, then it is easy to verify:

LEMMA 8. $\eta_{\mathbf{G}}$ is the unique Borel measure on \mathbf{G} such that $\widehat{\eta}_{\mathbf{G}}[\mathbf{1}] = 1$ and $\widehat{\eta}_{\mathbf{G}}[\chi] = 0$ for all other $\chi \in \widehat{\mathbf{G}}$. \square

More generally, we have the following:

LEMMA 9. Let $(\mathbf{G}, +)$ be a compact abelian group, and let $\mu \in \mathfrak{M}_{\text{cas}}(\mathbf{G})$. Then $(\mu = \eta_{\mathbf{S}} \text{ for some closed subgroup } \mathbf{S} \subseteq \mathbf{G}) \iff (\forall \chi \in \widehat{\mathbf{G}}, \text{ either } \widehat{\mu}[\chi] = 0 \text{ or } \widehat{\mu}[\chi] = 1).$

Proof: “ \implies ” Suppose $\mu = \eta_{\mathbf{S}}$. If $\chi \in \widehat{\mathbf{G}}$, then $\chi|_{\mathbf{S}} \in \widehat{\mathbf{S}}$. Thus, Lemma 8 implies that

$$\widehat{\mu}[\chi] = \widehat{\mu}[\chi|_{\mathbf{S}}] = \begin{cases} 1 & \text{if } \chi|_{\mathbf{S}} \equiv \mathbf{1}_{\mathbf{S}}; \\ 0 & \text{otherwise.} \end{cases}$$

“ \impliedby ” If $\gamma \in \widehat{\mathbf{G}}$ and $\widehat{\mu}[\gamma] = 1$, then $\text{supp}(\mu) \subseteq \ker(\gamma)$. If $\mathbf{S} := \bigcap \{ \ker(\gamma) ; \gamma \in \widehat{\mathbf{G}}, \widehat{\mu}[\gamma] = 1 \}$, then \mathbf{S} is a closed subgroup of \mathbf{G} , and $\text{supp}(\mu) \subseteq \mathbf{S}$. We claim $\mu = \eta_{\mathbf{S}}$. If $\chi \in \widehat{\mathbf{S}}$, then $\chi = \gamma|_{\mathbf{S}}$ for some $\gamma \in \widehat{\mathbf{G}}$ (this follows from the Pontrjagin Duality Theorem; see e.g. Fact (6), §0.7, p.13 of [Wal82]). If $\chi \neq \mathbf{1}_{\mathbf{S}}$, then $\mathbf{S} \not\subseteq \ker(\gamma)$, so $\widehat{\mu}[\gamma] \neq 1$, so $\widehat{\mu}[\gamma] = 0$ (by hypothesis), so $\widehat{\mu}[\chi] = 0$. Thus, Lemma 8 implies that $\mu = \eta_{\mathbf{S}}$. \square

Let $(\mathcal{A}, +)$ be a finite abelian group, so that $(\mathcal{A}^{\mathbb{M}}, +)$ is compact abelian. For any $\chi \in \widehat{\mathcal{A}^{\mathbb{M}}}$, there is a unique finite subset $\mathbb{B} \subset \mathbb{M}$ and unique nontrivial $\chi_{\mathbf{b}} \in \widehat{\mathcal{A}}$ for all $\mathbf{b} \in \mathbb{B}$ such that

$$\chi(\mathbf{a}) = \prod_{\mathbf{b} \in \mathbb{B}} \chi_{\mathbf{b}}(a_{\mathbf{b}}), \quad \forall \mathbf{a} \in \mathcal{A}^{\mathbb{M}}. \quad (7)$$

We say that χ is *based* on \mathbb{B} . If $\{\chi^{\mathbf{h}}\}_{\mathbf{h} \in \mathbb{H}} \subset \widehat{\mathcal{A}^{\mathbb{M}}}$ is an \mathbb{H} -indexed collection of characters based on \mathbb{B} , and μ is \mathbb{H} -mixing, then equations (2) and (7) together imply

$$\lim_{n \rightarrow \infty} \widehat{\mu} \left[\prod_{\mathbf{h} \in \mathbb{H}} \chi^{\mathbf{h}} \circ \sigma^{-n\mathbf{h}} \right] = \prod_{\mathbf{h} \in \mathbb{H}} \widehat{\mu}[\chi^{\mathbf{h}}]. \quad (8)$$

Proof of Theorem 3(c). If $\mathbf{C} = \text{supp}(\mu)$, then Theorem 3(a) says \mathbf{C} is an \mathcal{R}_{Φ} -coset shift —i.e. $\mathbf{C} = \mathbf{c} + \mathbf{S}$, where \mathbf{S} is an \mathcal{R}_{Φ} -submodule shift. Let $\nu := \tau^{-c}(\mu)$; then $\text{supp}(\nu) = \mathbf{S}$. To show that $\mu = \eta_{\mathbf{C}}$, we must show that $\nu = \eta_{\mathbf{S}}$; we will do this with Lemma 9. For any $\chi \in \widehat{\mathcal{A}^{\mathbb{M}}}$, note that

$$|\widehat{\mu}[\chi]| \stackrel{(*)}{=} |\chi(\mathbf{c}) \cdot \widehat{\nu}[\chi]| = |\chi(\mathbf{c})| \cdot |\widehat{\nu}[\chi]| \stackrel{(\dagger)}{=} |\widehat{\nu}[\chi]|. \quad (9)$$

[Here, $(*)$ is because $\mu = \tau^c(\nu)$ and χ is a homomorphism, while (\dagger) is because $\chi[\mathbf{c}] \in \mathbb{T}$.] Now, let $\chi \in \widehat{\mathcal{A}^{\mathbb{M}}}$, and suppose $\widehat{\nu}[\chi] \neq 1$; we must show that $\widehat{\nu}[\chi] = 0$. Let k be large enough that \mathcal{R}_Φ is generated by $\{\varphi_h^{p^k}\}_{h \in \mathbb{H}}$. Let Γ be the multiplicative group generated by $\{\varphi_h^{p^k}\}_{h \in \mathbb{H}}$ (which are units, by hypothesis).

CLAIM 1: $\Gamma \subset \mathcal{R}_\Phi$.

Proof: By definition, \mathcal{R}_Φ contains all products of positive powers of $\{\varphi_h^{p^k}\}_{h \in \mathbb{H}}$; we must show it contains the negative powers as well. For all $h \in \mathbb{H}$, the unit $\varphi_h^{p^k}$ has finite multiplicative order, because \mathcal{R}_Φ is finite; thus, there is some $n \in \mathbb{N}$ such that $(\varphi_h^{p^k})^{-1} = (\varphi_h^{p^k})^n \in \mathcal{R}_\Phi$. Thus, \mathcal{R}_Φ contains all products of integer powers (positive or negative) of $\{\varphi_h^{p^k}\}_{h \in \mathbb{H}}$; hence $\Gamma \subset \mathcal{R}_\Phi$. \diamond Claim 1

If $\gamma \in \Gamma$, then $\chi \circ \gamma \in \widehat{\mathcal{A}^{\mathbb{M}}}$ (it is a composition of two homomorphisms). Define

$$M := \max_{\gamma \in \Gamma} |\widehat{\nu}[\chi \circ \gamma]|. \quad (10)$$

Thus, $|\widehat{\nu}[\chi]| \leq M$. We will show that $M = 0$.

CLAIM 2: $M < 1$.

Proof: (by contradiction). Γ is finite, so if $M = 1$, then there is some $\gamma \in \Gamma$ such that $|\widehat{\nu}[\chi \circ \gamma]| = 1$. Let $t := \widehat{\nu}[\chi \circ \gamma]$; then $t \in \mathbb{T}$, and $\text{supp}(\nu) \subseteq (\chi \circ \gamma)^{-1}\{t\}$. But $\mathbf{0} \in \text{supp}(\nu)$ (because $\text{supp}(\nu) = \mathbf{S}$ is a submodule), so $t = 1$; hence $\widehat{\nu}[\chi \circ \gamma] = 1$. Thus, $\nu[\ker(\chi \circ \gamma)] = 1$, hence $\mathbf{S} \subseteq \ker(\chi \circ \gamma)$, hence $\gamma(\mathbf{S}) \subseteq \ker(\chi)$.

Now $\gamma^{-1} \in \Gamma \subset \mathcal{R}_\Phi$ (by Claim 1), and \mathbf{S} is an \mathcal{R}_Φ -module (by Theorem 3(a)), so $\gamma^{-1}(\mathbf{S}) \subseteq \mathbf{S}$, so $\mathbf{S} \subseteq \gamma(\mathbf{S}) \stackrel{(*)}{\subseteq} \ker(\chi)$, where $(*)$ is by the previous paragraph. But then $\widehat{\nu}[\chi] = 1$, which contradicts the definition of χ .

By contradiction, we must have $M < 1$. \diamond Claim 2

By replacing χ with $\chi \circ \gamma$ for some $\gamma \in \Gamma$ if necessary, we can assume without loss of generality that $|\widehat{\nu}[\chi]| = M$. Then $|\widehat{\mu}[\chi]| = M$ also, by eqn.(9). Thus it suffices to evaluate $\widehat{\mu}[\chi]$. But if Φ has polynomial representation (4), then for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \widehat{\mu}[\chi] &:= \int_{\mathcal{A}^{\mathbb{M}}} \chi \, d\mu \stackrel{(a)}{=} \int_{\mathcal{A}^{\mathbb{M}}} \chi \, d(\Phi^{(p^k)}\mu) \stackrel{(b)}{=} \int_{\mathcal{A}^{\mathbb{M}}} \chi \circ \Phi^{(p^k)} \, d\mu \\ &\stackrel{(c)}{=} \int_{\mathcal{A}^{\mathbb{M}}} \chi \circ \left(\sum_{h \in \mathbb{H}} \varphi_h^{(p^k h)} \sigma^{(p^k h)} \right) \, d\mu \stackrel{(h)}{=} \int_{\mathcal{A}^{\mathbb{M}}} \prod_{h \in \mathbb{H}} \left(\chi \circ \varphi_h^{(p^k h)} \circ \sigma^{(p^k h)} \right) \, d\mu \\ &= \widehat{\mu} \left[\prod_{h \in \mathbb{H}} \chi \circ \varphi_h^{(p^k h)} \circ \sigma^{(p^k h)} \right]. \end{aligned} \quad (11)$$

[Here, (a) is because μ is Φ -invariant; (b) is a change of variables; (c) is by eqn.(5); and (h) is because χ is a homomorphism.] Thus,

$$\widehat{\mu}[\chi] \stackrel{(11)}{=} \lim_{k \rightarrow \infty} \widehat{\mu} \left[\prod_{h \in \mathbb{H}} \chi \circ \varphi_h^{(p^k h)} \circ \sigma^{(p^k h)} \right] \stackrel{(*)}{\leq} \limsup_{k \rightarrow \infty} \prod_{h \in \mathbb{H}} \widehat{\mu} \left[\chi \circ \varphi_h^{(p^k h)} \right], \quad (12)$$

where $(*)$ is by eqn.(8), because μ is \mathbb{H} -mixing and Γ is finite. Thus,

$$M = |\widehat{\mu}[\chi]| \stackrel{(12)}{\leq} \limsup_{k \rightarrow \infty} \prod_{h \in \mathbb{H}} \left| \widehat{\mu} \left[\chi \circ \varphi_h^{(p^k h)} \right] \right| \stackrel{(*)}{\leq} \lim_{k \rightarrow \infty} \prod_{h \in \mathbb{H}} M = M^H.$$

[Here, $(*)$ is by equations (9) and (10).] But $H \geq 2$, and Claim 2 says $M < 1$. Thus, $M = 0$. Thus, $|\widehat{\nu}[\chi]| = 0$.

This holds for any $\chi \in \widehat{\mathcal{A}^{\mathbb{M}}}$ with $\widehat{\nu}[\chi] \neq 1$, so Lemma 9 says $\nu = \eta_{\mathbf{S}}$; hence $\mu = \eta_{\mathbf{C}}$. \square

Constructing coset shifts. To enumerate all \mathbb{H} -mixing, Φ -invariant measures, Theorem 3(c) says it suffices to enumerate all \mathbb{H} -mixing, Φ -invariant coset shifts. But if $\mathbf{S} \subset \mathcal{A}^{\mathbb{M}}$ is a submodule shift, not every coset of \mathbf{S} is a coset *shift*. Indeed, let $\mathbf{c} \in \mathcal{A}^{\mathbb{M}}$, and for all $\mathbf{m} \in \mathbb{M}$, let $\mathbf{b}^{\mathbf{m}} := \sigma^{\mathbf{m}}(\mathbf{c}) - \mathbf{c}$. Then it is easy to check

$$\left(\text{The coset } \mathbf{c} + \mathbf{S} \text{ is a subshift} \right) \iff \left(\mathbf{b}^{\mathbf{m}} \in \mathbf{S} \text{ for all } \mathbf{m} \in \mathbb{M} \right). \quad (13)$$

We call $\mathbf{B} := \{\mathbf{b}^{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{M}}$ the *coboundary* of \mathbf{c} . More generally, an $\mathcal{A}^{\mathbb{M}}$ -valued *cocycle* is any \mathbb{M} -indexed collection $\{\mathbf{b}^{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{M}} \subseteq \mathcal{A}^{\mathbb{M}}$ such that $\mathbf{b}^{\mathbf{m}+\mathbf{n}} = \sigma^{\mathbf{n}}(\mathbf{b}^{\mathbf{m}}) + \mathbf{b}^{\mathbf{n}}$, for all $\mathbf{n}, \mathbf{m} \in \mathbb{M}$.

LEMMA 10. (a) If $\mathbf{c} \in \mathcal{A}^{\mathbb{M}}$, then its coboundary \mathbf{B} is a cocycle.

(b) \mathbf{c} is entirely determined by c_0 and \mathbf{B} , because $c_{\mathbf{m}} = c_0 + b_0^{\mathbf{m}}$, for all $\mathbf{m} \in \mathbb{M}$.

(c) Let $\mathbf{B} := \{\mathbf{b}^{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{M}} \subset \mathbf{S}$ be any cocycle, and let $c_0 \in \mathcal{A}$ be arbitrary. Define $\mathbf{c} := [c_{\mathbf{m}}]_{\mathbf{m} \in \mathbb{M}}$ according to part (b). Then \mathbf{B} is the coboundary of \mathbf{c} ,

Proof: (a) is straightforward. (b) is because $c_{\mathbf{m}} = (\sigma^{\mathbf{m}}(\mathbf{c}))_0 = (\mathbf{c} + \mathbf{b}^{\mathbf{m}})_0 = c_0 + b_0^{\mathbf{m}}$. For (c), let $\mathbf{m}, \mathbf{n} \in \mathbb{M}$. Then $[\sigma^{\mathbf{m}}(\mathbf{c}) - \mathbf{c}]_{\mathbf{n}} = c_{\mathbf{m}+\mathbf{n}} - c_{\mathbf{n}} := (c_0 + b_0^{\mathbf{m}+\mathbf{n}}) - (c_0 + b_0^{\mathbf{n}}) = b_0^{\mathbf{m}+\mathbf{n}} - b_0^{\mathbf{n}} = (\mathbf{b}^{\mathbf{m}+\mathbf{n}} - \mathbf{b}^{\mathbf{n}})_0 \stackrel{(*)}{=} \sigma^{\mathbf{n}}(\mathbf{b}^{\mathbf{m}})_0 = b_{\mathbf{n}}^{\mathbf{m}}$, where $(*)$ is the cocycle property. This holds for all $\mathbf{n} \in \mathbb{M}$, so $\sigma^{\mathbf{m}}(\mathbf{c}) - \mathbf{c} = \mathbf{b}^{\mathbf{m}}$. This holds for all $\mathbf{m} \in \mathbb{M}$, so \mathbf{B} is the coboundary of \mathbf{c} . \square

Lemma 10 and Fact (13) imply that, to construct a coset shift of \mathbf{S} , it suffices to construct an \mathbf{S} -valued cocycle. In particular, if $a \in \mathcal{A}$ such that $a^{\mathbb{M}} \in \mathbf{S}$, then we can get one \mathbf{S} -valued cocycle by defining $\mathbf{b}^{\mathbf{m}} := (m_1 + \dots + m_D)a^{\mathbb{M}}$, for all $\mathbf{m} = (m_1, \dots, m_D) \in \mathbb{M}$; in Lemma 10(c) in this case, $c_{\mathbf{m}} = c_0 + (m_1 + \dots + m_D)a$ for all $\mathbf{m} \in \mathbb{M}$. For example, if \mathbf{S} is as in Example 4, then $1^{\mathbb{M}} \in \mathbf{S}$. In this case, $\mathbf{b}^{(m,n)} = 0^{\mathbb{M}}$ if $m+n$ is even, and $\mathbf{b}^{(m,n)} = 1^{\mathbb{M}}$ if $m+n$ is odd. If $c_0 := 0$, and we define \mathbf{c} as in Lemma 10(b), then we get precisely the ‘checkerboard’ configuration of Example 4; thus $\mathbf{C} = \mathbf{c} + \mathbf{S}$ is a nontrivial coset shift, as previously claimed.

Extension to rings of squarefree characteristic. We say $m \in \mathbb{N}$ is *squarefree* if m has prime factorization $m = p_1 p_2 \dots p_J$, where p_1, \dots, p_J are distinct primes. If \mathcal{R} has *nonprime* characteristic, then eqn.(5) is no longer true. However, the next result allows us to reduce squarefree-characteristic LCA to the previous case of prime-characteristic LCA.

PROPOSITION 11. Let \mathcal{R} be a commutative ring with characteristic $m = p_1^{s_1} p_2^{s_2} \dots p_J^{s_J}$, where p_1, \dots, p_J are distinct primes. For all $j \in [1..J]$, let $q_j := m/p_j^{s_j}$, and let $\mathcal{I}_j := \{r \in \mathcal{R} ; q_j r = 0\}$. Then

(a) \mathcal{I}_j is an ideal of \mathcal{R} , and the quotient ring $\mathcal{R}_j := \mathcal{R}/\mathcal{I}_j$ has characteristic $p_j^{s_j}$.

(b) \mathcal{R} is isomorphic to the direct product $\tilde{\mathcal{R}} := \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \dots \oplus \mathcal{R}_J$, via the map $\mathcal{R} \ni r \mapsto (r^1, r^2, \dots, r^J) \in \tilde{\mathcal{R}}$, where for all $j \in [1..J]$, we define $r^j := (r + \mathcal{I}_j) \in \mathcal{R}_j$.

(c) Let \mathcal{A} be any \mathcal{R} -module. For all $j \in [1..J]$, let $\mathcal{B}_j := \mathcal{I}_j \mathcal{A} := \{ia ; i \in \mathcal{I}_j \text{ and } a \in \mathcal{A}\}$ (a submodule), and let $\mathcal{A}_j := \mathcal{A}/\mathcal{B}_j$ be the quotient module. Then $\mathcal{A} \cong \tilde{\mathcal{A}} := \mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_J$, via the map $\mathcal{A} \ni a \mapsto (a^1, \dots, a^J) \in \tilde{\mathcal{A}}$, where for all $j \in [1..J]$, we define $a^j := (a + \mathcal{B}_j) \in \mathcal{A}_j$. Furthermore, $\tilde{\mathcal{R}}$ acts on $\tilde{\mathcal{A}}$ componentwise; that is, for any $\tilde{r} = (r^1, \dots, r^J) \in \tilde{\mathcal{R}}$ and $\tilde{a} = (a^1, \dots, a^J) \in \tilde{\mathcal{A}}$, we have $\tilde{r} \cdot \tilde{a} = (r^1 a^1, \dots, r^J a^J)$.

Proof: (a) is straightforward. (b) follows from the Chinese Remainder Theorem for rings [DF91, Thm.17, p.268, §10.3] and the following claim.

CLAIM 1: [i] $\mathcal{I}_1 \cap \mathcal{I}_2 \cap \dots \cap \mathcal{I}_J = \{0\}$, and [ii] For any $j \neq k$, $\mathcal{I}_j + \mathcal{I}_k = \mathcal{R}$.

Proof: [i] If $r \in \bigcap_{j=1}^J \mathcal{I}_j$ then $q_j r = 0$ for all $j \in [1 \dots J]$. But $\gcd(q_1, \dots, q_J) = 1$, so Bezout's identity yields $z_1, \dots, z_J \in \mathbb{Z}$ such that $\sum_{j=1}^J z_j q_j = 1$. But then $r = 1 \cdot r = \sum_{j=1}^J z_j q_j r = \sum_{j=1}^J z_j 0 = 0$.

[ii] $\mathcal{I}_j + \mathcal{I}_k$ is an ideal, so it suffices to show that $\mathcal{I}_j + \mathcal{I}_k$ contains $1_{\mathcal{R}}$. Let $\tilde{\mathbf{Z}}_m$ be the subring of \mathcal{R} generated by $1_{\mathcal{R}}$; then $\tilde{\mathbf{Z}}_m$ is isomorphic to \mathbb{Z}/m . It is easy to check that $\tilde{\mathbf{Z}}_m \cap \mathcal{I}_j = p_j^{s_j} \tilde{\mathbf{Z}}_m$ and $\tilde{\mathbf{Z}}_m \cap \mathcal{I}_k = p_k^{s_k} \tilde{\mathbf{Z}}_m$. But then $1_{\mathcal{R}} \in p_j^{s_j} \tilde{\mathbf{Z}}_m + p_k^{s_k} \tilde{\mathbf{Z}}_m$, because $1 \in p_j^{s_j} \mathbb{Z}/m + p_k^{s_k} \mathbb{Z}/m$ by Bezout's identity, because $\gcd(p_j^{s_j}, p_k^{s_k}) = 1$ (because p_j and p_k are distinct primes). \diamond Claim 1

(c) follows from Claim 1 and the Chinese Remainder Theorem for modules [DF91, Ex.16-17, p.333, §10.3]. \square

EXAMPLE 12: Suppose $\mathcal{A} = \mathcal{R} := \mathbb{Z}/m$, where $m = p_1 \cdots p_J$. Then for all $j \in [1 \dots J]$, we have $\mathcal{I}_j = p_j \mathbb{Z}/m$ in Proposition 11(a), so $\mathcal{A}_j = \mathcal{R}_j = \mathbb{Z}/m / (p_j \mathbb{Z}/m) \cong \mathbb{Z}/p_j$ in Proposition 11(b,c) which is the classic Chinese Remainder Theorem. \diamond

Let $\mathcal{A} \cong \bigoplus_{j=1}^J \mathcal{A}_j$ as in Proposition 11(c). Define $\Psi : \mathcal{A}^{\mathbb{M}} \longrightarrow \bigoplus_{j=1}^J \mathcal{A}_j^{\mathbb{M}}$ by

$$\Psi([a_m]_{m \in \mathbb{M}}) := ([a_m^1]_{m \in \mathbb{M}}, [a_m^2]_{m \in \mathbb{M}}, \dots, [a_m^J]_{m \in \mathbb{M}}), \quad (14)$$

where, for any $a \in \mathcal{A}$, we write $a \cong (a^1, \dots, a^J)$ as in Proposition 11(c). Then Ψ is a σ -commuting, homeomorphic \mathcal{R} -module isomorphism. If $\mu \in \mathfrak{M}_{\text{cas}}(\mathcal{A}^{\mathbb{M}})$, then for all $j \in [1 \dots J]$, let μ_j be the projection of $\Psi(\mu)$ to $\mathcal{A}_j^{\mathbb{M}}$; we say that μ is a *joining* of μ_1, \dots, μ_J . (See e.g. [dlR06] or [Rud90, Ch.6] for more about joinings.)

PROPOSITION 13. *Let \mathcal{R} be a commutative ring of squarefree characteristic, let \mathcal{A} be an \mathcal{R} -module, and write $\mathcal{A} \cong \bigoplus_{j=1}^J \mathcal{A}_j$ as in Proposition 11(c). Let $\Phi \in \mathcal{R}\text{-LCA}(\mathcal{A}^{\mathbb{M}})$ have neighbourhood \mathbb{H} and coefficients $\{\varphi_h\}_{h \in \mathbb{H}}$, all of which are units. If $\mu \in \mathfrak{M}_{\text{cas}}(\mathcal{A}^{\mathbb{M}}; \Phi, \sigma)$ is \mathbb{H} -mixing, then μ is a joining of measures μ_1, \dots, μ_J , where for each $j \in [1 \dots J]$, μ_j is the Haar measure of some Φ_j -invariant \mathcal{R}_j -coset shift of $\mathcal{A}_j^{\mathbb{M}}$.*

Proof: For all $h \in \mathbb{H}$, write $\varphi_h = (\varphi_h^1, \dots, \varphi_h^J)$ as in Proposition 11(b). Then φ_h^j is a unit in \mathcal{R}_j for each $j \in [1 \dots J]$. For all $j \in [1 \dots J]$, let $\Phi_j \in \mathcal{R}_j\text{-LCA}(\mathcal{A}_j^{\mathbb{M}})$ have local rule $\phi(\mathbf{a}_{\mathbb{H}}^j) = \sum_{h \in \mathbb{H}} \varphi_h^j a_h^j$, for any $\mathbf{a}_{\mathbb{H}}^j \in \mathcal{A}_{\mathbb{H}}^j$. If Ψ is as in eqn.(14), then Proposition 11(c) implies that Ψ is a topological conjugacy from $(\mathcal{A}^{\mathbb{M}}, \Phi)$ to the direct product $(\mathcal{A}_1^{\mathbb{M}}, \Phi_1) \times \cdots \times (\mathcal{A}_J^{\mathbb{M}}, \Phi_J)$. For all $j \in [1 \dots J]$, let μ_j be the projection of $\Psi(\mu)$ to $\mathcal{A}_j^{\mathbb{M}}$; then $\mu_j \in \mathfrak{M}_{\text{cas}}(\mathcal{A}_j^{\mathbb{M}}; \Phi_j, \sigma)$ and is \mathbb{H} -mixing; hence, Theorem 3(c) implies that μ_j is the Haar measure for some \mathcal{R}_j -coset shift of $\mathcal{A}_j^{\mathbb{M}}$. \square

COROLLARY 14. *Let $\mathcal{A} = \mathbb{Z}/m$, where $m = p_1 \cdots p_J$ is squarefree. Let $\mathbb{M} := \mathbb{Z}^D \times \mathbb{N}^E$, and let $\Phi \in \mathbb{Z}/m\text{-LCA}(\mathcal{A}^{\mathbb{M}})$ have a neighbourhood of cardinality $H \geq 2$. Let $\mu \in \mathfrak{M}_{\text{cas}}(\mathcal{A}^{\mathbb{M}}; \Phi, \sigma)$ be (σ, H) -mixing, and suppose that either:*

[i] $D + E = 1$; or [ii] $h(\mu, \sigma) > \log_2(m) - \log_2(\min\{p_1, \dots, p_J\})$.

Then μ is a joining of the uniform Bernoulli measures on $\mathcal{A}_1^{\mathbb{M}}, \dots, \mathcal{A}_J^{\mathbb{M}}$, where $\mathcal{A}_j := \mathbb{Z}/p_j$ for each $j \in [1 \dots J]$.

Proof: Case [i] follows immediately from Proposition 13 and Case [i] of Corollary 5.

Case [ii]: For each $i \in [1 \dots J]$, let μ_i be as in Proposition 13. Then

$$\begin{aligned} h(\mu_i) &\underset{(*)}{\geq} h(\mu) - \sum_{i \neq j=1}^J h(\mu_j) \underset{(\dagger)}{\geq} h(\mu) - \sum_{i \neq j=1}^J \log_2(p_j) \stackrel{(\ddagger)}{=} h(\mu) - \log_2(m) + \log_2(p_i) \\ &\underset{(\diamond)}{>} \log_2(m) - \log_2(\min\{p_j\}_{j=1}^J) - \log_2(m) + \log_2(p_i) \geq 0. \end{aligned}$$

Here, $(*)$ is because $h(\mu) \leq \sum_{j=1}^J h(\mu_j)$, while (\dagger) is because for each $j \in [1 \dots J]$ we have $h(\mu_j) \leq h_{\text{top}}(\mathcal{A}_j^{\mathbb{M}}) = \log_2 |\mathcal{A}_j| = \log_2(p_j)$. Next, (\ddagger) is because $m = p_1 \cdots p_J$, and (\diamond) is by the hypothesis of Case [ii].

Thus, Case [ii] of Corollary 5 implies that μ_i is the uniform Bernoulli measure on $\mathcal{A}_i^{\mathbb{M}}$. This holds for all $i \in [1 \dots J]$; the result follows. \square

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